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LETTER TO THE EDITOR

Contra-universality in percolation with variable strength of connection

T Odagaki
Department of Physics, Brandeis Universality, Waltham, Massachusetts 02254-9110, USA

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Abstract. Strength of connection between two sites is introduced in a dynamic percolation process in spatially disordered systems, where the jump rate between two sites is assumed to have a power-law dependence on the distance between the sites with a cut-off. With use of the coherent medium approximation and the effective-z method by Odagaki and Lax, it is predicted that the conductivity exponent depends on the power, though the critical percolation density does not, that is, the percolation transition in this model is contra-universal.

The percolation theory introduced by Broadbent and Hammersley [1] has been applied to phenomena of enormous variety with a wide range of length scale [2]: formation of a spiral star cloud, earthquakes, population genetics, conduction in disordered media, nuclear physics, to name a few. The basic element of the theory is that a connection between two objects can be defined and that the question to be asked is whether there exists an infinitely extended network of the connected object in the system. The percolation theory is very versatile because of this simple definition of the problem. The connection between objects is always assumed to be either on or off so that geometrical clusters can be constructed without ambiguity. In contrast to this assumption, interaction between two points (corresponding to the connection) in physical systems usually depends on the distance between the points. (For example, the exchange interaction between two magnetic spins.) The assumption may be valid when the interaction is rather short range and it is the processes on crystal lattices we are concerned with. However, in processes on lattices with further neighbour interaction or in spatially disordered systems the assumption is far from acceptable. In this Letter, I introduce a generalised percolation process in which the connection between two objects depends on the distance between them. This process may be called *soft percolation.* As a representative process, I consider hopping transport in spatially disordered systems where the jump rate is assumed to depend on a power of the separation between two sites. I will predict that the percolation process is contrauniversal, i.e. the conductivity critical exponent depends on the power, but the critical percolation density does not.

Model. A carrier is assumed to perform hopping motion between sites distributed randomly with a given density. The jump rate $w_{r_1 r_2}$ of a carrier between sites at $\boldsymbol{r}_1$ and

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$\boldsymbol{r}_2$ is regarded as the connection between them and assumed to be

$$w_{r_1 r_2} \equiv w(r_{12}) = \begin{cases} w_0(1 - r_{12}/r_0)^\alpha & \text{if } r_{12} \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_2| < r_0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that a cut-off of the jump rate at r_0 is introduced and $\alpha = 0$ corresponds to the usual percolation on which many papers have been written [3, 4]. Percolation is judged by the DC diffusion constant $D(0)$:

$$D(0) = \lim_{u \to 0} -\frac{u^2}{2d} \sum_{\boldsymbol{r}} \langle (\boldsymbol{r} - \boldsymbol{r}_0)^2 \tilde{P}(\boldsymbol{r}, u|\boldsymbol{r}_0) \rangle \tag{2}$$

where d is the dimensionality, $\tilde{P}(\boldsymbol{r}, u|\boldsymbol{r}_0)$ is the Laplace transform of the conditional transition probability of a carrier from site $\boldsymbol{r}_0$ to site $\boldsymbol{r}$, u is the Laplace parameter and the angular bracket denotes the ensemble average [4].

The effective-z method. I employ the effective-z method based on the coherent medium approximation for spatially disordered systems [4]. (The coherent medium approximation is a generalised application of the idea of the coherent potential approximation to the random walk master equation which is assumed to govern the dynamics of the percolating carrier.) In this approximation, the DC diffusion constant is directly proportional to the coherent jump rate $w_c(0)$, which is determined by the self-consistency equation

$$\varepsilon I(\varepsilon, \eta, \alpha) = 1 - \frac{2}{z(\rho)} - \exp\left(-\frac{\eta}{z(\rho)}\right) \tag{3}$$

where $\varepsilon = [(z(\rho) - 2)/z(\rho)] w_c(0)/w_0$,

$$I(\varepsilon, \eta, \alpha) = \int_0^1 \frac{N(x)}{\varepsilon + (1 - x)^\alpha} \, \mathrm{d}x \tag{4}$$

$N(x)$ is the modified Hertz distribution

$$N(x) = \frac{d\eta x^{d-1}}{z(\rho)} \exp\left(-\frac{\eta x^d}{z(\rho)}\right) \tag{5}$$

and $\eta = \rho V_d(r_0)$ is the reduced density. ($V_d(r)$ is the volume of a d-dimensional sphere of radius r.) When (3) does not have a positive solution for $w_c(0)$, then $w_c(0)$ is identically zero. In (5), a $1/z(\rho)$ part of a solid angle around a given site is considered. The density-dependent parameter $z(\rho)$ is also used as the coordination number of the coherent system. Note that the parameter $z(\rho)$ can be varied continuously once the approximation scheme is established (for details, see [4]).

The parameter $z(\rho)$ is chosen in such a way that the diffusion constant $D(0)$ agrees with known results for the regular percolation $\alpha = 0$. When $\alpha = 0$, one can solve (3) without difficulty to find

$$w_c(0)/w_0 = (z(\rho)p - 2)/(z(\rho) - 2) \tag{6}$$

for $\rho > \rho_c$, where $p = 1 - \exp(-\eta/z(\rho))$ and ρ_c satisfies $z(\rho_c) = 2/[1 - \exp(-\eta_c/z(\rho_c))]$

with $\eta_c \equiv \rho_c V_d(r_0)$. I chose the parameter $z(\rho)$ so that

$$\frac{z(\rho)p-2}{z(\rho)-2}=\left(\frac{z(\rho_c)p-2}{z(\rho_c)-2}\right)^{t_d} \tag{7}$$

holds, where the critical percolation density ρ_c and the conductivity exponent t_d in d-dimensions are exploited from known results [3, 5]. It is straightforward to show from (6) and (7) that

$$w_c(0)/w_0 \sim (\rho-\rho_e)^{t_d} \tag{8}$$

when $\rho=\rho_c+0^+$.

Results. I am interested in the critical region where $w_e(0)$ or ε is very small.

When $\alpha \leqslant 0$, then $I(\varepsilon, \eta, \alpha)$ can be expanded in a Taylor series in ε. Thus the leading term of $w_c(0)$ behaves as $(\rho-\rho_c)^{t_d}$ for $\rho=\rho_c+0^+$.

When $\alpha > 0$, the Taylor expansion of $I(\varepsilon, \eta, \alpha)$ in ε is not possible and a careful analysis is necessary. However, by expanding $N(x)$ in a Taylor series around $x=1$, one can easily show that the leading term of $I(\varepsilon, \eta, \alpha)$ for small ε is given by

$$I(\varepsilon,\eta,\alpha)=\begin{cases} N(1)/(1-\alpha) & \text{when } 0<\alpha<1 \\ -N(1)\ln\varepsilon & \text{when } \alpha=1 \\ \dfrac{\pi N(1)}{\alpha\sin(\pi/\alpha)}\,\varepsilon^{1/\alpha-1} & \text{when } \alpha>1. \end{cases} \tag{9}$$

Noting the behaviour of (8), one finds that when $\rho=\rho_c+0^+$

$$w_c(0)\sim(\rho-\rho_c)^t \qquad t=\begin{cases} t_d & \text{when } \alpha<1 \\ \alpha t_d & \text{when } \alpha>1 \end{cases} \tag{10}$$

and

$$w_c(0)\sim-(\rho-\rho_c)^{t_d}/\ln(\rho-\rho_c) \qquad \text{when } \alpha=1. \tag{11}$$

When $\rho<\rho_c$, $w_c(0)$ vanishes identically. Therefore, the critical percolation density is not changed by α, but the conductivity exponent is modified for $\alpha \geqslant 1$. This is regarded as contra-universal.

In conclusion, I have shown that the percolation process can be generalised so as to include the strength of connection, which may be called a *soft* percolation process. I have studied the generalised process in spatially disordered systems with use of the coherent medium approximation and the effective-z method. The percolation transition in this process is considered to be contra-universal. The conductivity exponent depends on the parameter of the distance dependence of the jump rate, but the critical density does not depend on it. This result indicates that the usual percolation model will be useful to predict the critical point, but not good for the critical exponent. In fact there have been a number of works [6, 7] on the register network in which a certain fraction of bonds are insulating and the conductivity of conducting bonds has a distribution. Naturally, this model also yields the conductivity depending on the distribution [6, 7]. The effective-medium-type approximation yields incorrect critical exponents for regular register networks. However, it predicts the enhancement of the conductivity exponent when the distribution in the strength of the conducting bonds exists [6]. Therefore, the present prediction of the non-universal conductivity exponent is expected to be true, though the quantitative dependence might be confirmed by other means.

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